MEASURES OF AXIAL SYMMETRY FOR OVALS(1)

BY

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ABSTRACT

A measure of axial symmetry for ovals is defined, and eleven particular measures are studied. Lower bounds for these measures are determined on the classes of arbitrary ovals, centrally symmetric ovals, and ovals of constant breadth. The proofs of these results make use only of elementary geometry and the properties of convexity.

In this paper we shall deal with *ovals* in the Euclidean plane, i.e., with compact convex sets whose interior is non-void. Ovals can possess two kinds of symmetry: with respect to a point (central symmetry) and with respect to a line (axial symmetry). For simplicity, we shall refer to these as *centrality* and *axiality*, respectively, and shall say that an oval is *central* or *axial* if it possesses a center or an axis of symmetry. Our concern in this paper is with axiality, and in particular, it consists in "measuring" the degree to which an oval possesses this property. Corresponding properties and measures of centrality have been thoroughly reported by Grünbaum [4]. The only known results for axiality are found in papers of Nohl [6], Krakowski [5], and Chakerian and Stein [1].

Definition. A measure of axiality is a real-valued function f defined on the class κ of all ovals in E_2 and satisfying the following conditions:

(i) $0 \leqq f(K) \leqq 1$ for every oval K;

(ii) $f(K) = 1$ if and only if K is axial;

(iii) f is similarly-invariant, i.e., it assumes equal values on similar ovals.

A particular measure of axiality is defined by means of intuitive considerations based on the properties of symmetry and the geometry of convex sets. Of special interest is the determination of the minimum value of a measure of axiality and of an extremal oval on which this value is attained, not only on the class of all ovals, but also on certain subclasses, such as the central ovals and ovals of constant breadth.

It is worth pointing out that, by property (iii) of the preceding definition, it

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is sufficient to consider a measure of axiality as defined on the space κ_s of equivalence classes of similarity-equivalence ovals. Thus, for example, we may limit ourselves, with no loss in generality, to the set of ovals whose diameter is 1.

It has been pointed out by Grünbaum [4, p. 237] that κ_s is not a compact space. This implies, in particular, that a minimum value and an extremal oval for a measure of axiality defined on κ_s may not exist. In such a case, we seek to determine the greatest lower bound of the measure and a sequence of ovals on which this bound can be approximated as closely as desired.

With these preliminary remarks in mind, we will now define and investigate, in the following sections, eleven measures of axiality for ovals.

1. The set of midpoints of all chords of an oval K in a fixed direction is called the *load curve* ("schwerlinie") of K in the direction ϕ, and denoted $\lambda_\phi(K)$. The basic facts about load curves are contained in a paper of Zindler [8]. Here we make use of the elementary observations that every load curve of an oval is connected and that in the direction ϕ_d normal to a diameter of d of K, the endpoints of $\lambda_{\phi_d}(K)$ are also the endpoints of this diameter. This last remark follows from the well known fact that each support line to an oval K in the direction ϕ_d meets βK (the boundary of K) in only one point, viz., an endpoint of the diameter d.

Let $b_\phi(K)$ denote the breadth of K, and $b_\phi\{Cv[\lambda_\phi(K)]\}$ the breadth of the convex hull of $\lambda_\phi(K)$ in the direction ϕ. If K has an axis of symmetry in the direction $\phi + \pi/2$, then this axis contains $\lambda_\phi(K)$, and $b_\phi\{Cv[\lambda_\phi(K)]\}=0$. Conversely, if the load curve for a given direction is a line segment normal to this direction, this segment determines an axis of symmetry for K, since it bisects all chords of K in this direction. For an oval K, $b_\phi(K) > 0$ for every ϕ,

$$0 \leqq \frac{b_\phi\{Cv[\lambda_\phi(K)]\}}{b_\phi(K)} \leqq 1 ,$$

and equality on the left holds if and only if K has an axis of symmetry normal to ϕ. Since the set of all directions in the plane is compact, the maximum of

$$f_1(\phi, K) = 1 - \frac{b_\phi\{Cv[\lambda_\phi(K)]\}}{b_\phi(K)}$$

exists on this set, for a fixed K. From these remarks, it is clear that the function

$$f_1(K) = \max_\phi \{(f_1(\phi, K)\}$$

is a measure of axiality on the set of ovals in E_2.

THEOREM 1. *For every oval K, $f_1(K) > \frac{1}{2}$, and this bound is best possible.*

Before proceeding with the proof of this theorem, several lemmas are needed.

LEMMA 1. *For every oval K, there is a direction ϕ_0 for which $f_1(\phi_0, K) > \frac{1}{2}$.*

Proof. It is sufficient to take for ϕ_0 the direction ϕ_d (resp. ϕ_w) normal to a diameter (resp. *width*, i.e., a chord of K of minimal length) of K. (We shall prove the lemma only for the direction ϕ_d, since the proof in the other case is similar and can be accomplished with only minor changes.) To see this, let AB be any diameter of K, m_1 and m_2 the parallel support lines to K, and k_1 and k_2 the parallel support lines to $Cv[\lambda_{\phi_d}(K)]$ in the direction of AB (Fig. 1). First of all, since the points A and B are also the endpoints of $\lambda_{\phi_d}(K)$, $AB \subseteq Cv[\lambda_{\phi_d}(K)]$.

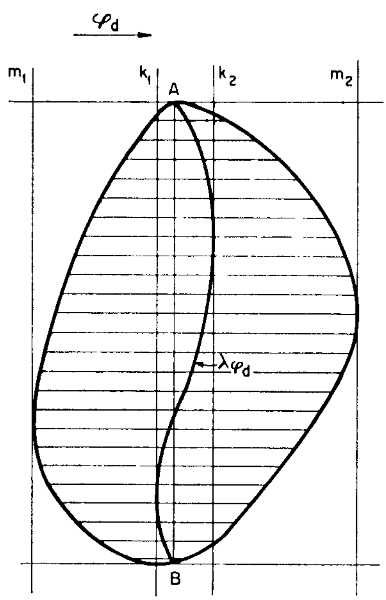

Fig. 1

Let x_1, x_2, x_3, x_4 be the distances, respectively, between the lines m_1 and k_1, k_1 and AB, AB and k_2, k_2 and m_2. Assume, by way of contradiction, that $f_1(\phi_d, K) < \frac{1}{2}$, i.e., that $x_1 + x_4 < \frac{1}{2} b_{\phi_d}(K)$. If both $x_2 \leqq x_1$ and $x_3 \leqq x_4$, then we have $b_{\phi_d}(K) = x_1 + x_2 + x_3 + x_4 \leqq 2(x_1 + x_4) < b_{\phi_d}(K)$, which is absurd. We may suppose, then, with no loss in generality, that $x_2 > x_1$. Since k_1 is a support line to $Cv[\lambda_{\phi_d}(K)]$, there is at least one point P of $b_{\phi_d}(K)$ on k_1. The inequality $x_2 > x_1$ implies that the chord of K in the direction ϕ_d having P as it midpoint does not meet the diameter AB, which is not possible for a convex set K. Therefore, our assumption is false, and $f_1(\phi_d, K) \geqq \frac{1}{2}$.

To prove Theorem 1, it remains to show that for some direction ϕ_0 we have the strict inequality $f_1(\phi_0, K) > \frac{1}{2}$.

LEMMA 2. *If* $f_1(\phi_d, K) = 1/2$, *where* ϕ_d *is normal to a diameter* $d = AB$ *of* K, $AB \subseteq \beta K$, *and conversely.*

Proof. From Lemma 1, it is not possible that $x_2 > x_1$ or $x_3 > x_4$; therefore, $f_1(\phi_d, K) = \frac{1}{2}$ implies $x_1 + x_4 = x_2 + x_3 = \frac{1}{2} b_{\phi_d}(K)$, and also that $x_1 = x_2$, and $x_3 = x_4$. If $x_1 = x_2 > 0$, then (as in the proof of Lemma 1) the midpoint of some chord of K in the direction ϕ_d lies on k_1, and one of its endpoints lies on AB^0, the open segment between A and B. Since the endpoint of a chord of K is in βK, this implies that a point of the open segment AB^0, and hence (by convexity) the entire closed segment AB, belongs to βK. In this case $x_3 = x_4 = 0$, and $AB \subseteq m_2$, which proves the first part of the lemma. The converse is trivial.

LEMMA 3. *If* d *is a diameter and* w *a width of an oval* K, *then at most one of these chords is a subset of* βK.

The proof of this lemma is a straightforward application of the basic properties of convexity.

LEMMA 4. *If* ϕ_d *and* ϕ_w *denote the directions normal to a diameter and a width, respectively, of an oval* K, *then either* $f_1(\phi_d, K) > \frac{1}{2}$ *or* $f_1(\phi_w, K) > \frac{1}{2}$.

Proof. By Lemma 1, $f_1(\phi_d, K) \geqq \frac{1}{2}$ and $f_1(\phi_w, K) \geqq \frac{1}{2}$. If $f_1(\phi_w, K) > \frac{1}{2}$, we are done, so we assume that $f_1(\phi_w, K) = \frac{1}{2}$. If $w = CD$ is a width of K, and $CD \subseteq \beta K$, then this lemma is an immediate consequence of the two preceding. Therefore, we may also assume that $CD^0 \subseteq K^0$, the interior of K. This situation, which indicates that Lemma 2 is not true if we replace "diameter" with "width," is possible, as the following example shows (Fig. 2). The segment CD is the only

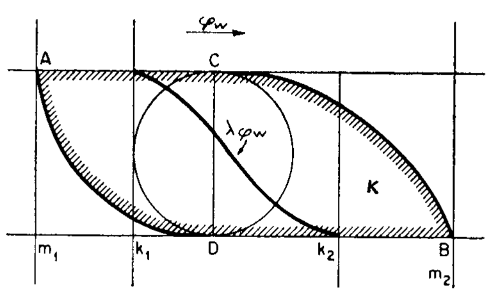

Fig. 2

width of K, which has been constructed so that the circle with CD as diameter intersects βK only in the two points C and D. The load curve λ_{ϕ_w} has for endpoints the midpoints of the segments AC and BD, which both lie in βK, so that $f_1(\phi_w, K) = \frac{1}{2}$ and yet $CD \nsubseteq \beta K$.

Let distances x_i, $i = 1, 2, 3, 4$, be defined as in Lemma 1, except that the four lines in question are now in the direction of the width CD of K. Since

$f_1(\phi_w, K) = \frac{1}{2}$, by hypothesis, then, as in the proof of Lemma 2, $x_1 = x_2$ and $x_3 = x_4$, there is at least one point of the width CD which is the endpoint of a chord of K in the direction ϕ_w, and hence at least one point of CD which belongs to βK. But since $CD^0 \subseteq K^0$, by hypothesis, such a point can only be an endpoint of CD. Hence, under our two assumptions that $f_1(\phi_w, K) = \frac{1}{2}$ and $CD^0 \subseteq K^0$, the points C and D must be endpoints of segments AC and BD in βK, where A and B are vertices of a supporting rectangle R of K with sides in the direction ϕ_w. Since $K \subseteq R$, $d(K) \leqq d(R) = AB$ ($d(S)$ denotes the diameter of the set S); since $A, B \in \beta K$, $AB \leq d(K)$; therefore, $AB = d(K)$, and $AB^0 \subseteq K^0$. By Lemma 2, this implies that $f_1(\phi_d, K) > \frac{1}{2}$, where ϕ_d is normal to AB. This completes the proof of the Lemma.

Proof of Theorem 1. The inequality $f_1(K) > \frac{1}{2}$ follows immediately from Lemma 4, so that it suffices to show that there exist ovals whose measure by f_1 is arbitrarily close to $\frac{1}{2}$, implying that this bound cannot be improved. Let ABC be a triangle, and ϕ_0 a direction in the plane which is neither parallel nor normal to the sides of ABC. Then the points A, B, C lie on distinct lines in this direction, and we may assume that A lies on a line between the other two. Let AD be the chord of ABC through the point A in the direction ϕ_0, and E its midpoint. Support lines to ABC in the direction $\phi_0 + \pi/2$ each meet it in a single vertex only, by the choice of ϕ_0, say in points A and C (Fig. 3). Then the load

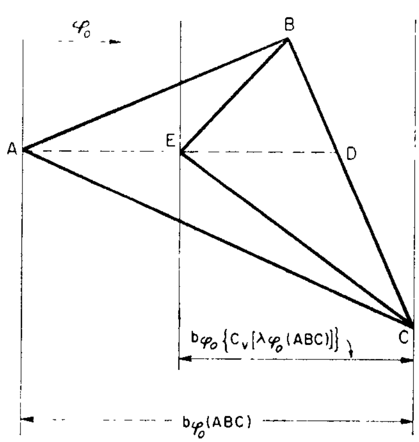

Fig. 3

curve $\lambda_{\phi_0}(ABC)$ is the polygonal arc BEC, and since D does not lie on the support line through C, $b_{\phi_0}\{Cv[\lambda_{\phi_0}(ABC)]\} > \frac{1}{2}b_{\phi_0}(ABC)$, and ϕ_0 cannot be a direction for which the maximum value of $f_1(\phi, ABC)$ is attained. Therefore, to determine this maximum, i.e. the value of the measure f_1, it is sufficient to consider the six (not necessarily distinct) directions parallel and normal to the sides of the triangle.

Now consider a right triangle ABC, with AB its diameter. Then it is easy to see that of these six directions, $f_1(\phi, ABC) > \frac{1}{2}$ for only one of these, viz. the direction ϕ_w normal to the width of ABC, which is the altitude to AB. By choosing angle B nearly a right angle, $f_1(\phi_w, ABC)$ can be made arbitrarily close to 1/2. If f_1 has a minimum value it can only be $\frac{1}{2}$, but in the light of Lemma 4, this minimum value is not attained on κ. Hence, the value $\frac{1}{2}$ is only the greatest lower bound of the measure f_1, and there is no extremal figure. This completes the proof of the theorem.

The best possible lower bound of f_1 on the subclass of central ovals is not known. However, we conjecture that it is $\sqrt{2}/2$, since this is the greatest lower bound of f_1 on the class of parallelograms, as can be seen from a direct computation of $f_1(\phi_w)$ and $f_1(\phi_d)$. The proof of this last statement will not be given here (See [2]).

Let $S = ABCD$ denote a unit square circumscribed about an oval K_1 of constant breadth 1, E and F the midpoints of AB and CD, respectively, and ζ the closed (shaded) region of Fig. 4, constructed by drawing circular arcs of radius

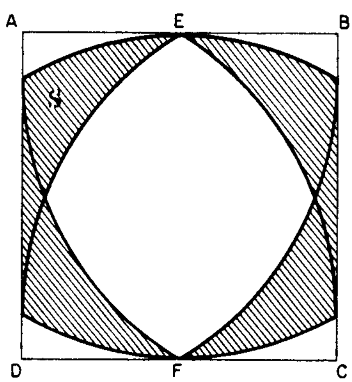

Fig. 4

1 about E and F, and about the intersections of these arcs with the other two sides. From properties of ovals of constant breadth, we may rotate K_1 inside S so that the points E and F belong to βK_1. For this position of K_1, which we shall call a *standard position* with respect to the square S, we have the following

LEMMA 5. $\beta K_1 \subseteq \zeta$.

The proof of this lemma is a consequence of elementary properties of ovals of constant breadth.

THEOREM 2. *For an oval K_1 of constant breadth, $f_1(K_1) \geqq \sqrt{\{2\sqrt{3}-3\}} \sim 0.681$.*

Proof. With K_1 in standard position with respect to a circumscribed square S, the points E and F of Fig. 5 are the endpoints of $\lambda_\phi(K_1)$, where ϕ is normal to EF. As a consequence of the lemma, a chord of K_1 in this direction must have a midpoint which is no farther from the diameter EF than the point X, the midpoint of the segment OP in this direction, where O is a vertex of the Reuleaux triangle

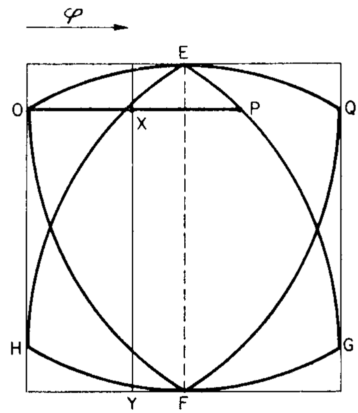

Fig. 5

OFQ, and P lies on the boundary of the Reuleaux triangle EGH. Since S is a unit square, an easy calculation yields $OP = \sqrt{\{2\sqrt{3} - 3\}}$, and from the above remarks we have $b_\phi\{Cv[\lambda_\phi(K_1)]\} \leqq 2 \cdot YF = 2(\frac{1}{2} - OX) = 2(\frac{1}{2} - \frac{1}{2}OP)$, and $f_1(K_1) \geqq f_1(\phi, K_1) = \sqrt{\{2\sqrt{3} - 3\}}$, since $b_\phi(K_1) = 1$ for every ϕ.

It is doubtful that this bound is best possible. We are assured, however, by the *Auswahlsatz* of Blaschke that a minimum value for f_1 on the subclass of ovals of constant breadth and an extremal figure exist, since a sequence of such ovals cannot converge to a degenerate limit, but only to another oval of constant breadth.

2. Let ϕ be a direction in the plane, m_1 and m_2 the support lines to an oval K in this direction, and $k = k(\phi)$ a line intersecting K and normal to ϕ. Let $k \cap m_1$ be the origin of a system of rectangular coordinates; then the support line m_2 passes through the point $b_\alpha(K)$ on the y-axis, where $\alpha = \phi + \pi/2$ (Fig. 6). For every $y \in [0, b_\alpha(K)]$, there is a chord $\gamma = \gamma_k(y)$ of K in the direction ϕ which is either divided by k into two parts $c = c_k(y)$ and $C = C_k(y)$, chosen such that $c/C \leqq 1$, or which does not meet k. We define

$$r(\phi, k, y) = \begin{cases} c/C, & \text{if } \gamma \cap k \neq \varnothing \\ 0, & \text{if } \gamma \cap k = \varnothing \end{cases}$$

If K is axial, then for some direction ϕ and some line k (which coincides with an axis of symmetry of K), $r(\phi, k, y) = 1$ for every $y \in [0, b_\alpha(K)]$, and conversely, if $r(\phi, k, y) = 1$ for every y and for a line k normal to the direction ϕ, then k is an axis of symmetry for K in the direction α. With these facts in mind, we can define a measure of axiality by considering the mean value of the ratios $r(\phi, k, y)$ over the interval $[0, b_\alpha(K)]$, then choosing the "best possible" line k in the fixed direction α, and finally, the "best possible" direction. More precisely, we define, for every oval K, the measure of axiality

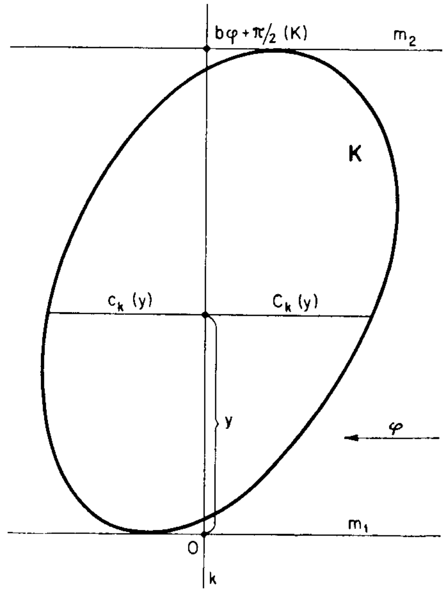

Fig. 6

$$f_2(K) = \max_{\phi} \max_{k} \left\{ \frac{1}{b_\alpha(K)} \int_0^{b_\alpha(K)} r(\phi, k, y) dy \right\}.$$

It is necessary to adopt the mean value of the ratios $r(\phi, k, y)$ rather than, say, the minimum value, since in this latter case the analogously defined measure has values arbitrarily close to zero on right triangles with one angle nearly zero, which renders it uninteresting.

THEOREM 3. *For every oval* K, $f_2(K) > \frac{1}{4}$.

Proof. We may assume that K has diameter $AB = 1$, so that it is contained in a lens-shaped region bounded by two arcs of radius 1 drawn about the points A and B as centers (Fig. 7). Let C and D be the points of intersection of βK with k, the perpendicular bisector of AB, and k_1 and k_2 the lines of support to K through the points A and B, respectively. If γ is any chord of K in the direction ϕ^d of AB, either γ meets k or not. If $\gamma \cap k \neq \emptyset$, $\gamma = FJ$, and $\gamma \cap k =$ H, then either $1 \geqq HJ / HF \geqq HI / HE$ or $1 \geqq HF / HJ \geqq HG / HK$ (see Fig. 7). Since $HI / HE = HG / HK$ ($ACBD$ is a symmetric quadrilateral), it is not necessary to distinguish these two cases. If m_1 and m_2 are lines through C and D, respectively, in the direction ϕ^d, then every chord of K lying between these lines meets k. The average value of the ratios HI/HE, for all such chords, is $\frac{1}{2}$. Therefore, if $b = b_{\phi^d+\pi/2}(K)$, and $h = CD$, it follows from these remarks that

$$f_2(K) \geqq \frac{1}{b} \int_0^b r(\phi^d, k, y) dy \geqq \tfrac{1}{2} \cdot \frac{h}{b}.$$

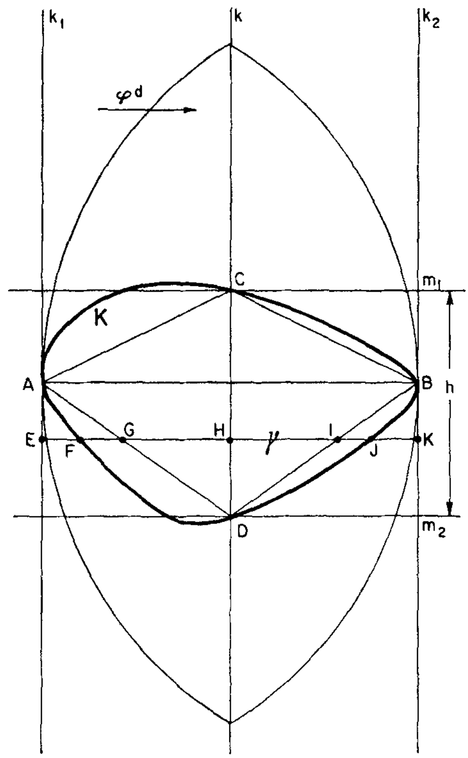

Fig. 7

The theorem is proved by showing that $h/b > \frac{1}{2}$. But this is an almost trivial consequence of convexity and the fact that K is contained in the given lens-shaped region.

THEOREM 4. *For a central oval* K_c, $f_2(K_c) \geqq 2\log 2 - 1 \sim 0.382$.

Proof. It is sufficient to consider the ratios $r(\phi, k, y)$ for those chords lying to one side of the center O of K_c, since the condition of centrality implies that these ratios for the two "halves" of the oval are identical. Let AO be half of a diameter AA' of K_c, and BC the chord passing through O and perpendicular to AO. Construct the isosceles trapezoid $T{=}BCC'B'$ by drawing $CC' \parallel BA$ and $BB' \parallel CA$. If K_c' is that "half" of K_c lying on the same side of BC as A, then clearly $K_c' \subseteq \mathrm{T}$, for were this not the case, a support line to K_c at B or C would intersect the interior of its other "half," which is not possible for a convex set. If $\gamma = EI$ is any chord of K_c' in the direction ϕ_d normal to AO, and $G = EI \cap AO$, then (referring to Fig. 8) either $1 \geqq EG/GI \geqq FG/GJ$ or $1 \geqq GI/GE \geqq GH/GD$. As in the preceding theorem, because of symmetry only one of these cases need be treated. Since f_2 is similarity-invariant, we may assume $OC = OB = 1$, and $OA = d$. Then the mean value of the ratios FG/GJ, for all chords of K_c' in the direction ϕ_d is given by

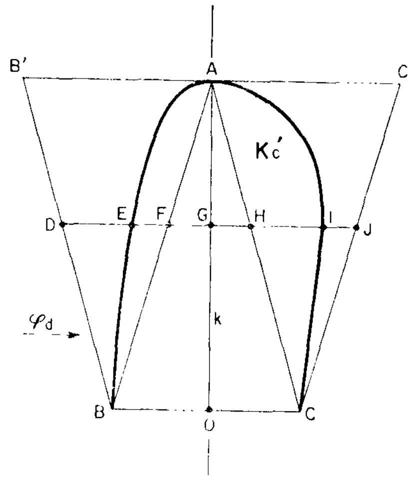

Fig. 8

$$\frac{1}{d}\int_0^d \frac{1-y/d}{1+y/d}\,dy = 2\log 2 - 1,$$

independently of d.

From the above remarks and inequalities, it is clear that

$$f_2(K_c) \geqq \frac{1}{b}\int_0^b r(\phi_d, k, y)dy \geqq 2\log 2 - 1, \text{ where } b = b_{\phi_d+\pi/2}(K_c).$$

THEOREM 5. *For an oval K_1 of constant breadth, $f_2(K_1) \geqq \alpha_0 \sim 0.5474$.*

Proof. We make use of Lemma 5, and consider K_1 to be in standard position with respect to a circumscribed square $S = ABCD$. If $E = \beta K_1 \cap AB$, $F = \beta K_1 \cap CD$, ϕ_d is the direction normal to $k = EF$, and $\gamma = XY$ is a chord of K_1 in the direction ϕ_d, then $XY \cap EF = \{Z\} \neq \varnothing$, and we may assume that $YZ/XZ \leqq 1$ (Fig. 9). Clearly, $YZ/XZ \geqq ZY'/ZX'$, where Y' lies on an arc of

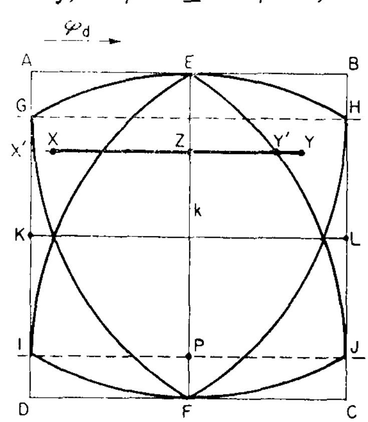

Fig. 9

the Reuleaux triangle EIJ, and X' lies either on a side of S or on an arc of the Reuleaux triangle FGH, depending on whether XY lies between the lines GH and KL (position 1) or between the lines GH and AB (position 2). By symmetry, similar observations can be made for chords lying between KL and CD.

Let P be the origin of a system of rectangular coordinates, and EF the y-axis. By symmetry, the mean value of the ratios ZY'/ZX' as functions of y is the same for $y \in [\sqrt{3}/2 - 1, \sqrt{3}/2 - \frac{1}{2}]$ as for $y \in [\sqrt{3}/2 - \frac{1}{2}, \sqrt{3}/2]$, so it is sufficient to consider only those chords lying in, say, the "upper" half square.

For $y \in [\sqrt{3}/2 - \frac{1}{2}, \sqrt{3}/2]$, $ZY' = ZY'(y) = \sqrt{\{1 - y^2\}} - \frac{1}{2}$;

for $y \in [\sqrt{3}/2 - \frac{1}{2}, \sqrt{3} - 1]$, $ZX' = ZX'(y) = \frac{1}{2}$ (position 1);

for $y \in [\sqrt{3} - 1, \sqrt{3}/2]$, $ZX' = \sqrt{\{1 - (y + 1 - \sqrt{3}/2)^2\}}$ (position 2).

Therefore,

$$
\begin{aligned}
f_2(K_1) &\geqq \int_0^1 r(\phi_d, k, y)dy = \int_{\sqrt{3}/2-1}^{\sqrt{3}/2} \frac{ZY(y)}{ZX(y)}dy \geqq 2\int_{\sqrt{3}/2-1/2}^{\sqrt{3}/2} \frac{ZY'(y)}{ZX'(y)}dy \\
&= 2\int_{\sqrt{3}/2-1/2}^{\sqrt{3}-1} \frac{\sqrt{\{1-y^2\}}-\frac{1}{2}}{\frac{1}{2}}dy + \int_{\sqrt{3}-1}^{\sqrt{3}/2} \frac{\sqrt{\{1-y^2\}}-\frac{1}{2}}{\sqrt{\{1-(y+1-\sqrt{3}/2)^2\}}}dy \\
&= 4\int_{\sqrt{3}/2-1/2}^{\sqrt{3}-1} [\sqrt{\{1-y^2\}}-\tfrac{1}{2}]dy \\
&\qquad + 2\int_{\sqrt{3}-1}^{\sqrt{3}/2} \frac{\sqrt{\{1-y^2\}}}{\sqrt{\{1-(y+1-\sqrt{3}/2)^2\}}}dy \\
&\qquad - \int_{\sqrt{3}-1}^{\sqrt{3}/2} \frac{dy}{\sqrt{\{1-(y+1-\sqrt{3}/2)^2\}}} \\
&= \alpha_0 \sim 0.4774 + 0.5936 - 0.5236 = 0.5474.
\end{aligned}
$$

3. If an oval K is reflected about a line k which meets its interior, then the intersection of K and its reflected image K_k is an axial oval contained in K, and the convex hull of the union of K and K_k is an axial oval containing K. It follows from the *Auswahlsatz* that, among all the axial ovals in K, there is at least one with maximal area, and among all the axial ovals containing K, there is at least one with minimal area. Furthermore, if K is axial, then the largest (smallest) axial oval contained in (containing) K is clearly K itself, and conversely. These considerations lead to the definitions of the following measures of axiality (where $[S]$ denotes the area of the set S):

$$f_3(K) = \max_{K'} \left\{\frac{[K']}{[K]} \quad : K' \text{ is axial and } K' \subseteq K \right\};$$

$$f_4(K) = \max_{K''} \left\{\frac{[K]}{[K'']} \quad : K'' \text{ is axial and } K \subseteq K'' \right\}.$$

In looking for the minimum values of these measures of axiality, it is natural to expect an extremal figure to be a polygon, since any oval can be approximated arbitrarily closely by a convex polygon, and since the measures assume nearly equal values on an oval and its approximating polygon (f_3 and f_4 are continuous on κ). Now if a polygon P is reflected about a line k through its interior to obtain the congruent reflected image P_k, then $P \cap P_k$ and $Cv(P \cup P_k)$ are both axial polygons, and hence the greatest (least) axial oval contained in (containing) P will again be a convex polygon. For these reasons, it is worthwhile to seek the largest (smallest) axial polygon of a given number of sides contained in (containing) an oval K. Besides being of use in finding lower bounds for the measures f_3 and f_4, questions of this type are interesting in themselves, and we have considered them at length elsewhere [3].

The inequality $f_3(K) \geqq 5/8$ was first established by Krakowski [5], and $f_4(K) \geq \frac{1}{2}$ is a trivial consequence of the existence of a circumscribed rectangle of area no greater than twice that of the given oval. Nohl [6] has shown that $f_3(K_c) \geq 2(\sqrt{2} - 1)$, and in [3] we have established the existence, about every central oval K_c, of a circumscribed axial octagon $0(K_c)$ such that

$$[K_c] \,/\, [0(K_c)] \geqq \sqrt{2}/2, \text{ so that } f_4(K_c) \geqq \sqrt{2}/2\,.$$

It is well known that in every oval K_1 of constant breadth 1 there is an inscribed circle $C(K_1)$ of diameter $d \geqq 2(3 - \sqrt{3}) / 3$, with equality only for T_1, the Reuleaux triangle of constant breadth 1. Therefore, using this fact, and the isoperimetric inequality, we obtain

$$f_3(K_1) \geqq \frac{[C(K_1)]}{[K_1]} \geqq \frac{[C(T_1)]}{[K_1]} \geqq \frac{[C(T_1)]}{[C_1]} = \frac{8}{3}(2 - \sqrt{3}) \sim 0.715,$$

where C_1 is the circle of diameter 1.

The convex axial set of least known area which contains every oval of diameter 1 is the octagon $\mathcal{O}$ of Fig. 10, which J. Pàl [7] constructed by removing the two

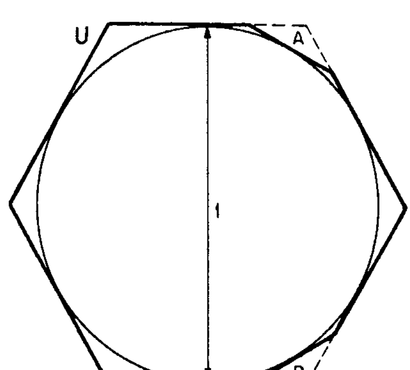

Fig. 10

isosceles triangles A and B tangent to the circle C_1 from alternate corners of the circumscribed regular hexagon. An elementary computation gives $[\mathscr{O}] = 2 - 2\sqrt{3}/3 \sim 0.8453$. Since $\mathscr{O}$ covers every oval K_1 of constant breadth 1, and is axial, we obtain the following inequality:

$$f_4(K_1) \geqq \frac{[K_1]}{[\mathscr{O}]} \geqq \frac{[T_1]}{[\mathscr{O}]} = \frac{3(\pi - \sqrt{3})}{4(3 - \sqrt{3})} \sim 0.8337.$$

4. Besides comparing the area of an oval with that of inscribed and circumscribed axial ovals, we may also compare their perimeters. This leads to to two measures of axiality analogous to f_3 and f_4, since it has been known from the time of Archimedes that if one convex curve is contained in another, the length of the former is strictly less than that of the latter. In particular, we define, for every oval K (where $|K|$ denotes the length of the curve βK) the measures of axiality

$$f_5(K) = \max_{K'} \left\{ \frac{|K'|}{|K|} : K' \text{ is axial and } K' \subseteq K \right\};$$

$$f_6(K) = \max_{K''} \left\{ \frac{|K|}{|K''|} : K'' \text{ is axial and } K \subseteq K'' \right\}.$$

As before, it is helpful in studying these measures, as well as being an interesting problem in itself, to investigate the bounds for perimeter ratios of inscribed and circumscribed axial polygons. This we have also done in [3], and shall make use of some of these results here. In particular, it is there shown that in every oval K there is an inscribed *kite* $Q(K)$, i.e. a quadrilateral symmetric about one of its diagonals, (which may degenerate to an isosceles triangle in a particular case) such that $|Q(K)|/|K| \geqq \beta_0 \sim 0.649$. From this it follows that $f_5(K) \geqq \beta_0$.

THEOREM 6. *For every oval* K, $F_6(K) \geqq \gamma_0 \sim 0.768$.

Proof. Assuming K has diameter $AB = 1$, it can be covered by a lens-shaped region, as in Fig. 11. Draw support lines EF and GH to K parallel to AB, and intersecting βK in the (not necessarily unique) points C and D, respectively. The triangles ACB and ADB (one of which may be degenerate if $AB \subseteq \beta K$) have minimum perimeter when they are isosceles, i.e. when C coincides with I, and D with J. Since the oval $FHGE$ is axial, we have

$$f_6(K) \geqq |K|/|FHGE| \geqq |ADBC|/|FHGE| \geqq |AJBI|/|FHGE|.$$

With B the origin of a rectangular coordinate system, and $OJ = x$, we define

$$a = AJ + BJ = \sqrt{\{4x^2 + 1\}}\ ;\ b = \widehat{AH} + HG + \widehat{GB} = 2 \arcsin x + 2\sqrt{\{1 - x^2\}} - 1\ ;$$

$$c = AI + IB\ ;\ d = \widehat{AF} + FE + \widehat{EB}.$$

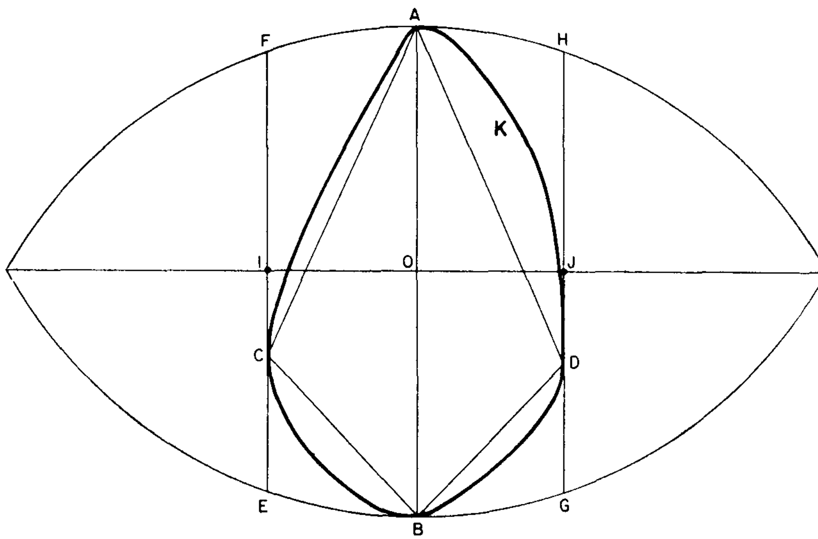

Fig. 11

By symmetry, it is sufficient to determine the minimum value of only one of the ratios a/b, c/d. For this purpose, let $f(x) = a/b$; f has a unique minimum value $\gamma_0 \sim 0.768$ on the interval $[0, \sqrt{3}/2]$ achieved for $x = x_0 \sim 0.327$. It follows from these remarks that

$$f_6(K) \geqq \frac{|AJBI|}{|FHGE|} \geqq \frac{a+c}{b+d} \geqq \min(a/b, c/d) = \min f(x) = \gamma_0$$

which completes the proof of the theorem.

In every central oval K_C there is an inscribed rhombus $R(K_c)$ such that $R(K_c) \mid / \mid K_c \mid \geqq \delta_0 \sim 0.8045$ [3]. Thus, $f_5(K_c) \geq \delta_0$. The same proof used to establish this fact also implies that $f_6(K_c) \geqq \delta_0$.

In every oval K_1 of constant breadth there is an inscribed kite $Q(K_1)$ such that $\mid Q(K_1) \mid / \mid K_1 \mid \geqq 2\sqrt{2}/\pi$ [3], which provides a lower bound for $f_5(K_1)$. The perimeter of the universal cover $\mathcal{O}$ of ovals of diameter 1 (Fig. 10) is

$$8(3 - \sqrt{3}) / 3 \sim 3.381,$$

as an easy computation shows, and since $\mathcal{O}$ is axial, we obtain the inequality

$$f_6(K_1) \geqq \frac{|K_1|}{|\mathcal{O}|} = \frac{3\pi}{8(3-\sqrt{3})} \sim 0.9291,$$

since every oval of constant breadth 1 has perimeter π.

We close this section with a few remarks relative to the two measures in question, and a conjecture. In every case where the extremal figure of the class κ for a measure of axiality or centrality is known, it is a triangle. To establish this fact for the measures of centrality analogous to f_5 and f_6, Grünbaum [4,

p. 257] makes use of a property of "superminimality." A measure f (of centrality or axiality) is said to possess this property if, for every pair of ovals K and K',

$$f(K + K') \geqq \min \{f(K), f(K')\},$$

where the "+" denotes Minkowski addition. In particular, if both K and K' are symmetric, $K + K'$ must be symmetric if this property is to hold. However, it is easy to see (Fig. 12) that the sum of two axial ovals need not be axial if their respective axes of symmetry are not parallel. Hence, no measure of axiality can possess this property, and it cannot be used to show that a triangle is an extremal figure for any such measure. Nevertheless, we conjecture that this is true for f_5 and have established the fact that $f_5(T) \geqq \eta_0 \sim 0.9168$, where T is a triangle [2]. The proof of this fact is elementary, but long, and we shall not give it here. If this conjecture is correct, then η_0 is the best possible lower bound for f_5.

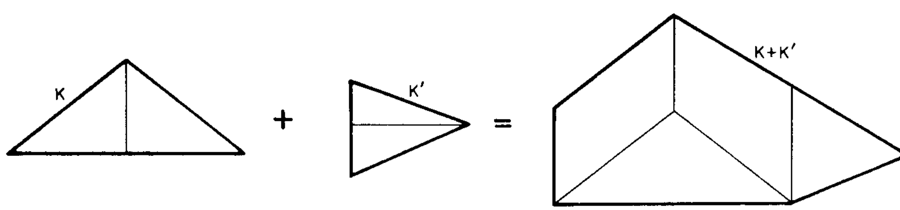

Fig. 12

5. Several measures of axiality arise from a method of symmetrization due to Steiner. The Steiner symmetrand $K^*_{k(\phi)}$ of an oval K with respect to a line k normal to the direction ϕ is an axial oval obtained by replacing every chord of K in the direction ϕ by a segment of equal length which lies along the same line and has its midpoint on $k = k(\phi)$.

If $k \cap K \neq \varnothing$, then $K \cap K^*_k \neq \varnothing$. If K is already axial, then it is clear that the Steiner symmetrand of K with respect to any of its axes of symmetry is K itself: if K is not axial, then for every direction ϕ, and for every line $k(\phi)$ normal to this direction, $K \cap K^*_{k(\phi)}$ is a proper subset of K, and $Cv[K \cup K^*_{k(\phi)}]$ properly contains K. It is well known that $[K^*_{k(\phi)}] = [K]$, and that $| K^*_{k(\phi)} | \leqq | K |$, with equality if and only if K is already axial with respect to a line parallel to $k(\phi)$. With these remarks in mind, we can define the following measures of axiality:

$$f_7(K) = \max_{\phi} \max_{k} \left\{ \frac{[K^*_{k(\phi)} \cap K]}{[K]} \right\};$$

$$f_8(K) = \max_{\phi} \max_{k} \left\{ \frac{[K]}{[Cv(K^*_{k(\phi)} \cup K)]} \right\};$$

$$f_9(K) = \max_{\phi} \max_{k} \left\{ \frac{|K^*_{k(\phi)} \cap K|}{|K|} \right\};$$

$$f_{10}(K) = \max_{\phi} \max_{k} \left\{ \frac{|K|}{|Cv(K^*_{k(\phi)} \cup K)|} \right\};$$

$$f_{11}(K) = \max_{\phi} \max_{k} \left\{ \frac{|K^*_{k(\phi)}|}{|K|} \right\}.$$

These measures are especially difficult to evaluate, even on the simplest ovals. However, lower bounds may be obtained for them from the facts that, for every oval K, $f_i(K) \geqq f_{i-4}(K)$, $i=7, 8, 9, 10$, and $f_{11}(K) \geqq f_9(K)$. Thus, the bounds we have obtained for the measures of axiality discussed in Sections 3 and 4 are also bounds for the measures of axiality defined in the present section. To establish these facts, we prove the following

THEOREM 7. *For every oval $K, f_7(K) \geqq f_3(K)$, with equality only when K is axial.*

Proof. In Fig. 13, K_k denotes the image of the oval K reflected in the line $k = k(\phi)$, and the broken curve is the boundary of $K^* = K^*_k$. Let $\bar{K}_k = K_k \cap K$

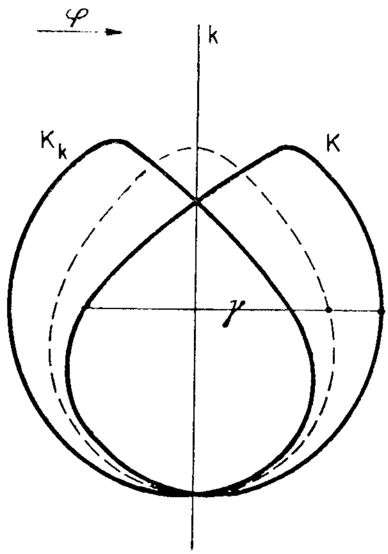

Fig. 13

and $\bar{K}^* = K^* \cap K$. If γ is a chord of K in the direction ϕ, then γ is either bisected by the line k or it is not. If it meets k but is not bisected by it, then γ is divided by k into two parts γ' and γ'', chosen such that $|\gamma'| < |\gamma''|$. Let $\bar{\gamma} = \gamma \cap \bar{K}_k$ and $\bar{\gamma}^* = \gamma \cap \bar{K}^*$; then

$$|\bar{\gamma}| = 2 \cdot |\gamma'| < |\gamma'| + |\gamma|/2 = |\bar{\gamma}^*| < |\gamma'| + |\gamma''| = |\gamma|,$$

by the definitions of $\bar{K}_k$ and $\bar{K}^*$. If γ is bisected by k, then $|\gamma'| = |\gamma''|$ and $|\bar{\gamma}| = |\bar{\gamma}^*| = |\gamma|$. If γ does not meet k at all, then $\bar{\gamma} = \varnothing$, and $|\bar{\gamma}^*| \leqq |\gamma|/2$. Thus, in all cases, we have $|\bar{\gamma}| \leqq |\bar{\gamma}^*| \leqq |\gamma|$. Furthermore, since this is true for every chord γ in the given direction ϕ, for every line $k(\phi)$ normal to this direction, and for every direction ϕ, we have in every case $\bar{K}_{k(\phi)} \subseteq \bar{K}^*_{k(\phi)} \subseteq K$, and $\bar{K}_{k(\phi)} = \bar{K}^*_{k(\phi)} = K$ only when $\bar{\gamma} = \bar{\gamma}^* = \gamma$, for every chord γ in a given direction, which can occur only if K is axial. From the definitions of f_3 and f_7, it now follows that $f_7(K) \geq f_3(K)$ for every oval K, with equality only when K is axial.

The other results stated in the preceding paragraph may be proved in a similar manner. We can improve the bound for f_{11} on the class of ovals of constant breadth by means of the following

THEOREM 8. *For ovals of constant breadth,* $f_{11}(K_1) \geqq \sqrt{\{2 - 2\sqrt{3}/\pi\}} \sim 0.948.$

Proof. Among all ovals of constant breadth 1 (and hence constant perimeter π) the Releaux triangle T_1 has least area and the circle C_1 the greatest area. Hence among all curves K^1 of constant breadth with fixed area 1, the Reuleaux triangle T^1 has the greatest, and the circle C^1 the least, perimeter. An easy calculation yields $|C^1| = 2\sqrt{\pi}$ and $|T^1| = \pi\sqrt{\{2/(\pi - \sqrt{3})\}}$. Since the area of an oval is invariant under Steiner symmetrization, a Steiner symmetrand $(K^1)^*$ of K^1 still has area 1, although it need no longer be an oval of constant breadth. From these remarks, the isoperimetric inequality, and the fact that Steiner symmetrization cannot increase the perimeter of an oval, it follows that

$$2\sqrt{\pi} = |C^1| \leqq |(K^1)^*| \leqq |K^1| \leqq |T^1| = \pi\sqrt{\{2/(\pi - \sqrt{3})\}},$$

and hence that

$$f_{11}(K^1) \geqq \frac{|(K^1)^*|}{|K^1|} \geqq \frac{|C^1|}{|T^1|} \geqq \sqrt{\{2 - 2\sqrt{3}/\pi\}}.$$

Since f_{11} is similarity-invariant, the same inequality holds for an oval of constant breadth 1.

In closing, it seems worthwhile to point out that only the result of Theorem 1 and that of Nohl [6] are certainly the best possible, and it is highly improbable that any of the others are. A paper of Chakerian and Stein [1] contains everything that seems to be known about analogous measures of symmetry in higher dimensions.

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